

On the Stability of Coherent States for Pais-Uhlenbeck Oscillator

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Abstract

We have constructed coherent states for the higher derivative Pais-Uhlenbeck Oscillator. In the process we have suggested a novel way to construct coherent states for the oscillator having only negative energy levels. These coherent states have negative energies in general but their coordinate and momentum expectation values and dispersions behave in an identical manner as that of normal (positive energy) oscillator. The coherent states for the Pais-Uhlenbeck Oscillator have constant dispersions and a modified Heisenberg Uncertainty Relation. Moreover, under reasonable assumptions on parameters these coherent states can have positive energies.

Introduction: The Pais-Uhlenbeck Oscillator (PUO) [1] can act as a prototype toy model of higher derivative (covariant) theories of gravity [2] since both are plagued by ghost excitations. In PUO the ghost problem is manifested in two complimentary ways: (i) The Hamiltonian can be bounded below but the system will contain negative norm states; (ii) the system will consist only of positive norm states but the Hamiltonian is not bounded below. In [3, 4] the authors solve both the problems by treating PUO as a non-Hermitian PT -symmetric quantum system. However some questions have been raised in [5, 6, 7] and in [7] an alternative consistent formulation of PUO has also been suggested. We follow the spirit of Smilga [5, 6] who suggests that the ghost induced non-unitarity is not a serious problem in the context of PUO since the theory is free. Indeed an important question is what will happen in presence of interactions. We will comment on this at the end. However

we point out that interactions will lead to an extended version of PUO. We are not dealing with this non-minimal form of PUO at present. In the present paper we follow the option (ii) stated above. We concentrate on the stability problem arising from the presence of negative energy levels. Specifically we show that coherent states, constructed in the Generalized Coherent State (GCS) framework, pioneered by Klauder, Gazeau and others [8, 9, 10], are stable with positive energy under reasonable assumptions on the parameters. In the process we have suggested a simple but novel way of accommodating systems with non-positive energy spectrum in GCS scheme. Recently we have applied GCS scheme [11] in extended Harmonic Oscillator models satisfying Generalized Uncertainty Principle and in non-linear oscillator model [12].

Pais-Uhlenbeck Oscillator in Hamiltonian framework: The dynamics of classical PUO is governed by the following equation of motion:

$$z^{(4)} + (\Omega^2 + \omega^2)z^{(2)} + \Omega^2\omega^2z = 0, \quad (1)$$

where $z^{(k)}$ denotes k -th time derivative of the real dynamical variable z and Ω, ω and positive real parameters. In the present work we will concentrate on the non-degenerate case $\Omega > \omega$ where $e^{\pm i\Omega t}$ and $e^{\pm i\omega t}$ are the solutions of (1). The dynamics can be obtained from the higher derivative Lagrangian,

$$L = \frac{1}{2}[\ddot{z}^2 - (\Omega^2 + \omega^2)\dot{z}^2 + \omega^2\Omega^2z^2], \quad (2)$$

or its classically equivalent counterpart,

$$L = \frac{1}{2}[\dot{q}^2 - (\Omega^2 + \omega^2)q^2 + \omega^2\Omega^2z^2] + \lambda(\dot{z} - q). \quad (3)$$

The Lagrangian (3) is quadratic at the expense of an additional real variable q besides z . This alternative is suitable for us. Let us follow Dirac's Hamiltonian formalism for constraint systems [13]. For the PUO this is discussed in [14].

The conjugate momenta $p_q = (\partial L)/(\partial \dot{q}) = \dot{q}$, $p_z = (\partial L)/(\partial \dot{z}) = \lambda$, $p_\lambda = (\partial L)/(\partial \dot{\lambda}) = 0$ show that there are two Second Class Constraints (non-commuting in the Poisson Bracket

sense) $\psi_1 = p_z - \lambda \approx 0$, $\psi_2 = p_\lambda \approx 0$ with $\{\psi_1, \psi_2\} = -1$. All the variables q, z, λ are considered to be canonical and independent. One can impose the constraints strongly in further analysis provided Dirac Brackets are used in place of Poisson Brackets. However, in the present system it is trivial to see that imposition of the constraints will reduce the set of variables to q, p_q, z, p_z and their Dirac Brackets will be identical to their original canonical Poisson Brackets. That is q, z continue to be independent and canonical.

The canonical Hamiltonian, with the constraints imposed, is,

$$\begin{aligned} H &= \dot{q}p_q + \dot{z}p_z + \dot{\lambda}p_\lambda - L \\ &= \frac{1}{2}[p_q^2 + 2qp_z + (\Omega^2 + \omega^2)q^2 - \omega^2\Omega^2z^2]. \end{aligned} \quad (4)$$

The effect of the constraints is seen in the crossterm qp_z in H (4) and still later in (32) where we explicitly demonstrate that the Coherent State constructed here does indeed satisfy the constraints.

Canonical transformation to free system of two oscillators: H is decoupled by exploiting the linear canonical transformation,

$$\begin{aligned} X &= \frac{p_z + \Omega^2 q}{\Omega\sqrt{\Omega^2 - \omega^2}}, \quad x = \frac{p_q + \Omega^2 z}{\sqrt{\Omega^2 - \omega^2}}, \\ P &= \frac{\Omega(p_q + \omega^2 z)}{\sqrt{\Omega^2 - \omega^2}}, \quad p = \frac{p_z + \omega^2 q}{\sqrt{\Omega^2 - \omega^2}}, \end{aligned} \quad (5)$$

yielding,

$$H = \frac{1}{2}(P^2 + \Omega^2 X^2) - \frac{1}{2}(p^2 + \omega^2 x^2). \quad (6)$$

Quantization of the individual oscillators leads to the energy spectrum $\Omega(n_1 + \frac{1}{2}) - \omega(n_2 + \frac{1}{2})$ that is clearly not positive definite due to the presence of the ω oscillator. However both the oscillators live in a positive norm Hilbert space. On the other hand one can have an equivalent and alternative setup where the ω oscillator will yield positive energy states but the states will have negative norm. These two scenarios result from the inherent ghost problem of higher derivative theories, PUO being an example. These issues are discussed in detail in [3, 4] where it is stressed that both choices need to be avoided. However, as commented

in [5, 6], the absence of a lower bound in energy does not cause a serious problem unless non-linear interactions are involved.

Coherent state for negative energy oscillator: Our subsequent analysis is in agreement with conclusions of [5, 6]. We will show that Coherent States for PUO can represent stable states with sensible and unambiguous dispersions, even though the energy may not be positive definite. In fact the Coherent State energy can be positive for the non-degenerate case studied here with some natural choice of parameters. We start by providing a brief description of the GCS for positive energy Harmonic Oscillator. Quite obviously generalization of Coherent States is not required here but we retain the notations of GCS as in [10]. The Harmonic Oscillator Hamiltonian is,

$$H = \frac{P^2}{2M} + \frac{M\Omega^2 X^2}{2}. \quad (7)$$

The Fock space is defined by

$$A | n \rangle = \sqrt{n} | n-1 \rangle, \quad A^\dagger | n \rangle = \sqrt{n+1} | n+1 \rangle. \quad (8)$$

H is diagonalized to

$$H = \Omega(A^\dagger A + \frac{1}{2}), \quad (9)$$

by the creation (annihilation) operator A^\dagger (A),

$$A = \sqrt{\frac{M\Omega}{2}}X + i\frac{P}{\sqrt{2M\Omega}}, \quad A^\dagger = \sqrt{\frac{M\Omega}{2}}X - i\frac{P}{\sqrt{2M\Omega}}, \quad [A, A^\dagger] = 1. \quad (10)$$

The GCS is defined as [10]

$$| J, \Gamma \rangle = \frac{1}{N(J)} \sum_{n=0}^{\infty} \frac{J^{(n/2)} e^{-i\Gamma E_n}}{\sqrt{R_n}} | n \rangle, \quad R_n = E_1 E_2 \dots E_n, \quad E_n = n \quad (11)$$

where J is related to energy and $\Gamma \sim \Omega t$ [10]. The normalization condition yields $\frac{1}{N(J)^2} \sum_{n=0}^{\infty} \frac{J^n}{R_n} = 1$. Expectation values of the dynamical variables X, P are computed easily by using the relations,

$$X = \frac{A + A^\dagger}{\sqrt{2M\Omega}}, \quad P = -i\sqrt{\frac{M\Omega}{2}}(A - A^\dagger). \quad (12)$$

This yields, for example,

$$\begin{aligned} \langle J, \Gamma | A | J, \Gamma \rangle &= \frac{1}{N^2(J)} \sum_{n,m=0}^{\infty} \frac{J^{(n+m)/2} e^{-i\Gamma(E_m - E_n)}}{\sqrt{R_m R_n}} \langle m | A | n \rangle \\ &= \sqrt{J} e^{-i\Gamma}, \end{aligned} \quad (13)$$

leading to,

$$\langle X \rangle = \sqrt{\frac{2J}{M\Omega}} \cos \Gamma, \quad \langle P \rangle = -\sqrt{2M\Omega J} \sin \Gamma. \quad (14)$$

Finally we come to the coherent state construction for negative energy oscillator. This is a new approach not considered before. Recall that we will opt for the second alternative scheme (ii) the system will consist only of positive norm states but the Hamiltonian is not bounded below. Hence we suggest that the same construction can be applied for the negative energy Hamiltonian,

$$h = -\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right), \quad (15)$$

with

$$a = \sqrt{\frac{m\omega}{2}} x + i \frac{p}{\sqrt{2m\omega}}, \quad a^\dagger = \sqrt{\frac{m\omega}{2}} x - i \frac{p}{\sqrt{2m\omega}}, \quad [a, a^\dagger] = 1, \quad (16)$$

leading to

$$h = -\omega \left(a^\dagger a + \frac{1}{2}\right). \quad (17)$$

Once again the Fock space is

$$a | n \rangle = \sqrt{n} | n-1 \rangle, \quad a^\dagger | n \rangle = \sqrt{n+1} | n+1 \rangle. \quad (18)$$

We propose the GCS to be,

$$| j, \gamma \rangle = \frac{1}{N(j)} \sum_{n=0}^{\infty} \frac{j^{(n/2)} e^{-i\gamma(-e_n)}}{\sqrt{\rho_n}} | n \rangle, \quad \rho_n = e_1 e_2 \dots e_n = (-1)^n R_n, \quad e_n = -n. \quad (19)$$

The degrees of freedom are

$$x = \frac{a + a^\dagger}{\sqrt{2m\omega}}, \quad p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger). \quad (20)$$

Expectation values of a, a^\dagger are different from the normal case (13),

$$\langle j, \gamma | a | j, \gamma \rangle = \frac{1}{N^2(J)} \sum_{n,m=0}^{\infty} \frac{j^{(n+m)/2} e^{i\Gamma(e_m - e_n)}}{\sqrt{\rho_m \rho_n}} \langle m | a | n \rangle$$

$$= -i\sqrt{j}e^{i\gamma} \quad (21)$$

leading to

$$\langle x \rangle = \sqrt{\frac{2j}{m\omega}} \sin \gamma, \quad \langle p \rangle = -\sqrt{2m\omega j} \cos \gamma. \quad (22)$$

These can be contrasted with (14). *It is interesting to note that the 90 degree phase shift between (22) and (14) is reminiscent of the 90 degree rotation to implement the complexification of one of the coordinates in [3, 4].*

One finds the dispersions and the uncertainty relation,

$$\langle x^2 \rangle = \frac{1}{2m\omega}(1 + 4j \sin^2 \gamma), \quad \langle p^2 \rangle = \frac{m\omega}{2}(1 + 4j \cos^2 \gamma), \quad (23)$$

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2m\omega}, \quad (\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{m\omega}{2} \\ (\Delta x)^2(\Delta p)^2 = 1/4. \quad (24)$$

For the normal case similar well known relations are,

$$\langle X^2 \rangle = \frac{1}{2M\Omega}(1 + 4J \cos^2 \Gamma), \quad \langle P^2 \rangle = \frac{M\Omega}{2}(1 + 4J \sin^2 \Gamma), \quad (25)$$

$$(\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2 = \frac{1}{2M\Omega}, \quad (\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2 = \frac{M\Omega}{2}, \\ (\Delta X)^2(\Delta P)^2 = 1/4. \quad (26)$$

The above are some of our major results indicating that as far as stability of the coherent states are considered, an oscillator with only negative energy levels behaves similarly as a normal positive energy oscillator.

Identical equations of motion satisfy the Correspondence Principle,

$$\langle \ddot{X} \rangle = -\Omega^2 \langle X \rangle, \quad \langle \ddot{x} \rangle = -\omega^2 \langle x \rangle \quad (27)$$

Finally energies of the respective GCS for normal and ghost oscillators are,

$$\langle H \rangle = \frac{\langle P^2 \rangle}{2M} + \frac{M\Omega^2 \langle X^2 \rangle}{2} = \frac{\Omega}{2}(1 + 2J), \\ \langle h \rangle = -\left(\frac{\langle p^2 \rangle}{2M} + \frac{m\omega^2 \langle x^2 \rangle}{2}\right) = -\frac{\omega}{2}(1 + 2j). \quad (28)$$

Coherent state for Pais-Uhlenbeck Oscillator: Since PUO is a combination of two non-interacting positive and negative energy oscillator (6), it is natural to consider the coherent state as a direct product of $|J, \Gamma\rangle$ and $|j, \gamma\rangle$, coherent states of the positive and negative energy oscillator respectively. At the same time, from (1) recall that our true concern should be with the z, p_z variables. These are related to X, P, x, p by the inverse transformations of (5):

$$\begin{aligned} z &= \frac{x - (P/\Omega)}{\sqrt{\Omega^2 - \omega^2}}, \quad p_z = \frac{\Omega^2 p - \Omega \omega^2 x}{\sqrt{\Omega^2 - \omega^2}}, \\ q &= \frac{\Omega X - p}{\sqrt{\Omega^2 - \omega^2}}, \quad p_q = \frac{\omega P - \omega^2 x}{\sqrt{\Omega^2 - \omega^2}}. \end{aligned} \quad (29)$$

Hence it is straightforward to compute $\langle z \rangle$,

$$\begin{aligned} \langle z \rangle &= \frac{1}{\sqrt{\Omega^2 - \omega^2}} (\langle x \rangle - \frac{\langle P \rangle}{\Omega}) \\ &= \frac{1}{\sqrt{\Omega^2 - \omega^2}} (\sqrt{\frac{2J}{\Omega}} \sin \Gamma + \sqrt{\frac{2j}{\omega}} \sin \gamma). \end{aligned} \quad (30)$$

Utilizing the operator relation

$$\dot{z} = \frac{1}{\sqrt{\Omega^2 - \omega^2}} (-p + \Omega X), \quad (31)$$

one can check that the GCS satisfies the constraint $\lambda(\dot{z} - q) \sim 0$ in (3):

$$\langle \dot{z} \rangle = \frac{1}{\sqrt{\Omega^2 - \omega^2}} (\sqrt{2\omega j} \cos \gamma + \sqrt{2\Omega J} \cos \Gamma) = \langle q \rangle. \quad (32)$$

In the last equality we have used (29).

Next we calculate the dispersion $(\Delta z)^2$,

$$\langle z^2 \rangle = \frac{1}{(\Omega^2 - \omega^2)} \left[\frac{1}{2} \left(\frac{1}{\omega} + \frac{1}{\Omega} \right) + 2(\sqrt{(j/\omega)} \sin \gamma + \sqrt{(J/\Omega)} \sin \Gamma_1) \right]^2, \quad (33)$$

$$(\Delta z)^2 = \langle z^2 \rangle - \langle z \rangle^2 = \frac{1}{2\Omega\omega(\Omega - \omega)}, \quad (34)$$

as well as the dispersion $(\Delta p_z)^2$,

$$\langle p_z \rangle = -\frac{1}{\sqrt{\Omega^2 - \omega^2}} (\Omega^2 \sqrt{2\omega j} \cos \gamma + \omega^2 \sqrt{2\Omega J} \cos \Gamma), \quad (35)$$

$$\langle p_z^2 \rangle = \frac{1}{(\Omega^2 - \omega^2)} \left[\frac{1}{2} \Omega \omega (\omega^3 + \Omega^3) + 2(\omega^2 \sqrt{\Omega J} \cos \Gamma + \Omega^2 \sqrt{\omega j} \cos \gamma)^2 \right], \quad (36)$$

$$(\Delta p_z)^2 = \frac{\Omega \omega (\omega^2 + \Omega^2 - \omega \Omega)}{2(\Omega - \omega)}. \quad (37)$$

Hence the modified Heisenberg Uncertainty Relation for the physical z, p_z variables is revealed:

$$(\Delta z)^2 (\Delta p_z)^2 = \frac{(\omega^2 + \Omega^2 - \omega \Omega)}{4(\Omega - \omega)^2}. \quad (38)$$

Discussions and Conclusion: There are several interesting points and peculiarities to be noticed in the behavior of the PU Oscillator variable z . Due to the transformations (29) the dimensions of z, p_z are different from their counterparts X, P or x, p . Furthermore due to the coordinate-momentum mixing in the transformations (29), the profiles of $\langle z \rangle, \langle p_z \rangle, \langle z^2 \rangle, \langle p_z^2 \rangle$ in (30,33,36) are quite involved with the parameters mixed up, as compared to the corresponding forms of for a ghost or normal HO $\langle x \rangle, \langle p \rangle, \langle X \rangle, \langle P \rangle$ in (23-26). However, things get miraculously cleared up once the dispersions $(\Delta z)^2, (\Delta p_z)^2$ are computed in (34,37). Since the relations are independent of $\gamma = \omega t, \Gamma = \Omega t$, the dispersions are time invariant, similar to normal Harmonic Oscillator. This indicates stability of the GCS.

For $\Omega \gg \omega$, we find,

$$(\Delta z)^2 \sim \frac{1}{\omega \Omega^2} + \frac{1}{\Omega^3}, \quad (\Delta p_z)^2 \sim \omega \Omega^2. \quad (39)$$

The above immediately provides the leading order correction in the Uncertainty Relation,

$$(\Delta z)^2 (\Delta p_z)^2 \sim \frac{1}{4} \left(1 + \frac{\omega}{\Omega} \right). \quad (40)$$

The energy of the GCS for PU Oscillator will be

$$E = \frac{\Omega}{2} (1 + 2J) - \frac{\omega}{2} (1 + 2j). \quad (41)$$

Since the parameter J or j can be identified with $|z|^2$ [10] it is probably natural to consider $J = j$. In that case the GCS energy is positive for $\Omega > \omega$ that is being assumed here.

Finally we comment on the question regarding the stability of the system of two *interacting* oscillators having positive and negative energy levels. Indeed, generically the system will be unstable but as Smilga [5, 6] has shown that there are certain specific form of interactions for which the system is stable. However any interaction term will clearly lead to a non-minimal form of Pais-Uhlenbeck Oscillator which is not our concern in the present work. However it will be interesting to see how effect of interactions is reflected in the coherent states that we have constructed here. We expect to report on this in near future.

To conclude, we have studied the non-degenerate version of the Pais-Uhlenbeck Oscillator. We suggest that Generalized Coherent States are probably better suited to deal with the Pais-Uhlenbeck Oscillator. From previous works [3, 4, 5, 6, 7] it is clear that in spite of the presence of negative energy ghost states the system can be subjected to a consistent quantization program. Our system lives entirely in positive norm Hilbert space but we allow presence of negative energy states and hence the vacuum is unbounded from below. Hence our main concern is the stability and energy positivity of the coherent states. We have precisely established that the coherent states constructed here for Pais-Uhlenbeck Oscillator can have positive energy under reasonable assumptions on the parameters and the states have constant coordinate and momentum dispersions ensuring their stability.

References

- [1] A. Pais and G. E. Uhlenbeck, Phys. Rev. 79, 145 (1950).
- [2] K. S. Stelle, Phys. Rev. D 16, 953 (1977).
- [3] C. M. Bender and P. D. Mannheim, Phys. Rev. Lett. 100, 110402 (2008);
- [4] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
- [5] A. V. Smilga, SIGMA, 5, 017(2009);

- [6] Phys. Lett. B 632, 433 (2006).
- [7] Ali Mostafazadeh Phys. Lett. A375: 93 (2010) (arXiv:1008.4678).
- [8] J.R.Klauder, J.Math.Phys. 4 (1963) 1055;
- [9] J.R.Klauder and J.-P.Gazeau, J.Phys. A32 (1999) 123.
- [10] J.-P. Antoine, J. -P. Gazeau, P. Monceau, J.R. Klauder and K.A. Penson, J.Math.Phys. 42 (2001) 2349.
- [11] S. Ghosh, P. Roy, Phys.Lett. B711, (2012) 423 (arXiv:1110.5136).
- [12] S. Ghosh, Journal of Mathematical Physics 53 (2012) (arXiv:1111.7129).
- [13] P.A.M. Dirac, Lectures on Quantum Mechanics, (Yeshiva University Press, New York, 1964).
- [14] P. D. Mannheim and A. Davidson, Phys. Rev. A 71, 042110 (2005).